Zadatak 1 *S točnošću većom od 10^{-6} odredite $\cos 710°$. Izračunajte grešku.*

Rješenje.

$$\cos 710° = \cos(2 \cdot 360° - 10°) = \cos 10° = \cos\frac{\pi}{18}$$

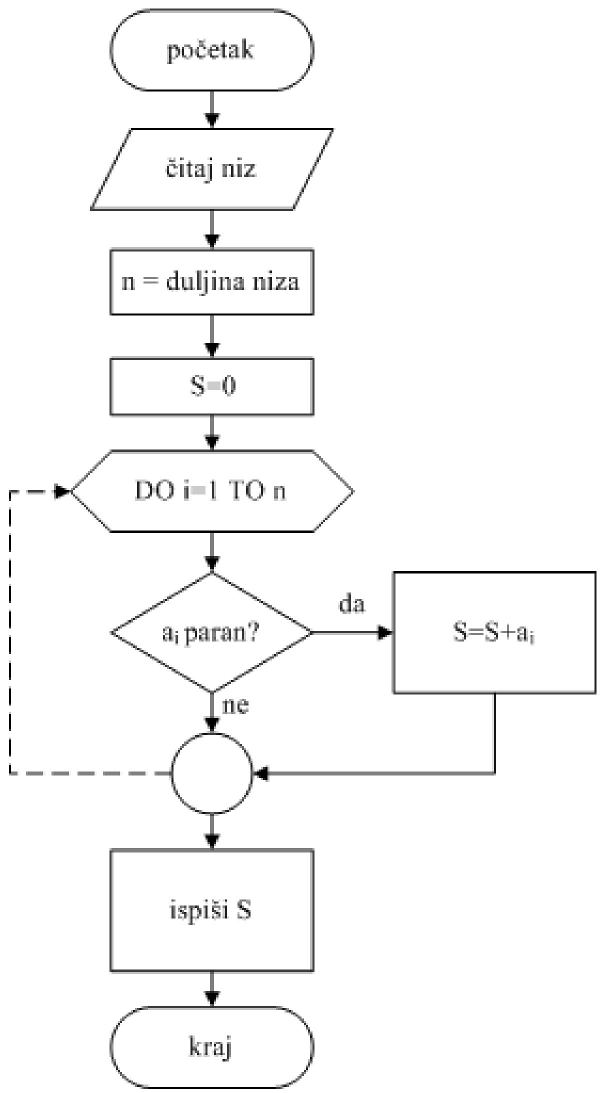
$$n = 2 \Rightarrow R_6\left(\frac{\pi}{18}\right) \leq \frac{\left(\frac{\pi}{18}\right)^6}{6!} = 0.39 \cdot 10^{-7} < 0.25 \cdot 10^{-6}$$

$$\Rightarrow \cos\frac{\pi}{18} = 1 - \frac{\left(\frac{\pi}{18}\right)^2}{2!} + \frac{\left(\frac{\pi}{18}\right)^4}{4!} = 1 - 0.0152309 + 0.0000387 = 0.9848078$$

$$\varepsilon = 0.39 \cdot 10^{-7} + 2 \cdot 0.5 \cdot 10^{-7} + 0 = 0.139 \cdot 10^{-6} < 10^{-6}.$$

Zadatak 2 *Opišite dijagram toka i napišite program u Mathematica-i za algoritam koji za zadani niz duljine n računa sumu parnih članova tog niza.*

Rješenje.




```
A = { , , , ... , , };
n = Length[A];
S = 0;
For[i = 1, i ≤ n, i = i + 1,
  If[IntegerQ[A[[i]]/2] == True, S = S + A[[i]]];
 ];
Print[S]
```

Slika 1:

Zadatak 3 *Jacobijevom metodom (jednom iteracijom) odredite približno rješenje sustava*

$$\begin{aligned} 3x_1 + x_2 &= 5 \\ 2x_1 + 6x_2 &= 9. \end{aligned}$$

Rješenje.

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\Rightarrow x^{(0)} = D^{-1}b = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 3/2 \end{bmatrix}$$

$$\Rightarrow x^{(1)} = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 5 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 5/3 \\ 3/2 \end{bmatrix}\right) = \begin{bmatrix} 7/6 \\ 17/18 \end{bmatrix}$$

Pravo rješenje:

$$\begin{aligned} 3x_1 + x_2 &= 5 \\ 2x_1 + 6x_2 &= 9 \end{aligned} \Leftrightarrow \begin{aligned} -18x_1 - 6x_2 &= -30 \\ 2x_1 + 6x_2 &= 9 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 21/16 \\ 17/16 \end{bmatrix}$$

Prava greška:

$$\varepsilon = \sqrt{(7/6 - 21/16)^2 + (17/18 - 17/16)^2} = 0.18763$$

Zadatak 4 *Metodom najmanjih kvadrata odredite vezu oblika* $bx - ay = xy$ *ako je*

x_k	-1	1	2
y_k	1.1	-1.7	-4.8

.

Rješenje.

$$\frac{1}{y} = \frac{1}{b} + \frac{a}{b}\frac{1}{x} \Rightarrow \bar{y} = a_0 + a_1\bar{x}, \quad \bar{y} = \frac{1}{y}, \bar{x} = \frac{1}{x}, a_0 = \frac{1}{b}, a_1 = \frac{a}{b}$$

$\bar{x}_i$	-1	1	0.5
$\bar{y}_i$	0.91	-0.59	-0.21

$$\Rightarrow \sum_{i=0}^{2}\bar{x}_i = 0.5, \sum_{i=0}^{2}\bar{x}_i^2 = 2.25, \sum_{i=0}^{2}\bar{x}_i\bar{y}_i = -1.605, \sum_{i=0}^{2}\bar{y}_i = 0.11 \Rightarrow a_0 = 0.1615, a_1 = -0.7492$$

$$\Rightarrow b = \frac{1}{a_0} = 6.19, \; a = a_1 b = 4.64 \Rightarrow 6.19x + 4.64y = xy$$

Zadatak 5 *Odredite polinom prvog stupnja koji metodom najmanjih kvadrata aproksimira funkciju* $f(x) = \sin x$ *na intervalu* $[0, \pi/2]$.

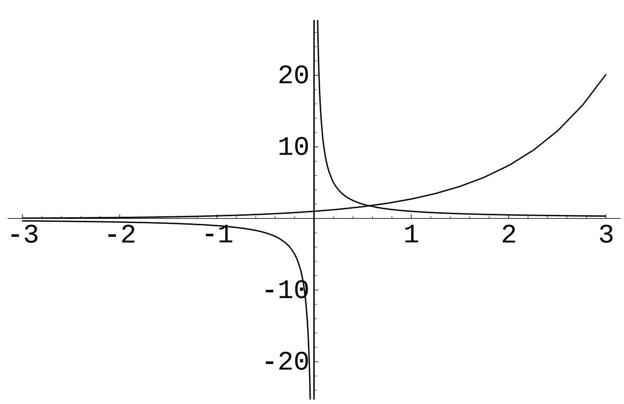
Rješenje.

$$\int_0^{\pi/2} x dx = \frac{\pi^2}{8}, \quad \int_0^{\pi/2} x^2 dx = \frac{\pi^3}{24}, \quad \int_0^{\pi/2} \sin x dx = 1, \; \int_0^{\pi/2} x \sin x dx = 1$$

$$a_0 = 0.11477, \quad a_1 = 0.66444 \Rightarrow \varphi(x) = 0.11477 + 0.66444x$$

Zadatak 6 *Pripremite za Newtonovu metodu i izračunajte prvu aproksimaciju rješenja za jednadžbu* $e^x = \frac{1}{x}$.

Rješenje.



Slika 2

$$f(x)=e^x-\frac{1}{x},\quad f(0.1)=-8.89<0,\quad f(0.9)=1.35>0 \Rightarrow \text{ nultočka je unutar intervala } [0.1,0.9]$$

$$f'(x)=e^x+\frac{1}{x^2}>0 \Rightarrow m_1=f'(0.9)=3.69 \Rightarrow f''(x)=e^x-\frac{2}{x^3}<0 \Rightarrow M_2=f''(0.1)=1998.89$$

$$\Rightarrow 270.85(x_n-x_{n-1})^2<\varepsilon$$

$$x_0=0.1 \Rightarrow x_1=0.1-\frac{f(0.1)}{f'(0.1)}=0.18798$$

Zadatak 7 *Koristeći opći Newtonov oblik interpolacijskog polinoma odredite interpolacijski polinom za funkciju* $f(x)=\ln x$ *i čvorove* $x_0=1,\ x_1=1.05,\ x_2=1.1$. *Izračunajte* $\ln 1.04$, *te uniformnu, lokalnu i pravu grešku.*

Rješenje.

x_i	y_i	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$
$x_0=1$	$y_0=0$		
		$f[x_0,x_1]=0.9758$	
$x_1=1.05$	$y_1=0.04879$		$f[x_0,x_1,x_2]=-0.454$
		$f[x_1,x_2]=0.9304$	
$x_2=1.1$	$y_2=0.09531$		

$$L_2(x)=0.9758(x-1)-0.454(x-1)(x-1.05)=-0.454x^2+1.9065x-1.4525$$

Kako je sada $L_2(1.04)=0.03976$ i $\ln 1.04=0.03922$ za pravu grešku imamo $|L_2(1.04)-\ln 1.04|=0.537\cdot 10^{-3}$.

$$f'(x)=\frac{1}{x} \Rightarrow f''(x)=-\frac{1}{x^2} \Rightarrow f'''(x)=\frac{2}{x^3} \Rightarrow M_3(x)=\max_{x\in[1,1.1]}\left|\frac{2}{x^3}\right|=2$$

Uniformna greška: $\frac{0.05^3}{4\cdot 3}\cdot 2=0.208\cdot 10^{-4}$

Lokalna greška: $\frac{2}{6}|(1-1.04)(1.05-1.04)(1.1-1.04)|=0.8\cdot 10^{-5}$.

Zadatak 8 *Za funkciju* $f(x)=\sqrt[3]{2+x}$ *poznate su vrijednosti* $f(1)$, $f(1.5)$ *i* $f(2)$. *Odredite* $f'(1.5)$:

a) Hermiteovom metodom ako je još poznato i $f'(2)$,

b) koristeći kubni splajn ako su poznate vrijednosti $f'(1)$ *i* $f'(2)$,

c) numeričkim diferenciranjem.

Izračunajte pravu grešku u sva tri slučaja.

Rješenje. a) $f'(x)=\frac{1}{3\sqrt[3]{(2+x)^2}}$

x_i	y_i	$f^{[1]}$	$f^{[2]}$	
$x_{-1}=1.5$	$y_{-1}=1.51829$			
		$f'(x_{-1})=?$		
$x_{-1}=1.5$	$y_{-1}=1.51829$		$f[x_{-1},x_{-1},x_0]=?$	
		$f[x_{-1},x_0]=0.15208$		$f[x_{-1},x_{-1},x_0,x_1]=?$
$x_0=1$	$y_0=1.44225$		$f[x_{-1},x_0,x_1]=-0.01386$	
		$f[x_0,x_1]=0.14515$		$f[x_{-1},x_0,x_1,x_1]=0.00198$
$x_1=2$	$y_1=1.5874$		$f[x_0,x_1,x_1]=-0.01287$	
		$f'(x_1)=0.13228$		
$x_1=2$	$y_1=1.5874$			

$$\frac{-0.01386-f[x_{-1},x_{-1},x_0]}{-0.5}=0.00198 \Rightarrow f[x_{-1},x_{-1},x_0]=-0.01287$$

$$\Rightarrow \frac{0.15208-f'(x_{-1})}{-0.5}=-0.01287 \Rightarrow f'(0.5)=0.14565.$$

Kako je prava vrijednost $f'(0.5)=0.144599$ za pravu grešku imamo $|0.14565-0.144599|=0.104\cdot 10^{-2}$.

b)

x_i	y_i	$f[x_i, x_{i+1}]$
$x_0 = 1$	$y_0 = 1.44225$	
		$f[x_0, x_1] = 0.15208$
$x_1 = 1.5$	$y_1 = 1.51829$	
		$f[x_1, x_2] = 0.13822$
$x_2 = 2$	$y_2 = 1.5874$	

$$\Rightarrow 0.5s_0 + 2s_1 + 0.5s_2 = 3(0.5(0.15208) + 0.5(0.13822)) = 0.43545.$$

Kako je $s_0 = f'(1) = 0.16025$ i $s_2 = f'(2) = 0.13228$ imamo $s_1 = 0.14459$, a prava grška je $|0.14459 - 0.144599| = 0.65 \cdot 10^{-6}$.

c)

$$f'(1.5) = \frac{1}{2 \cdot 0.5}(f(2) - f(1)) = 0.14515$$

Prava greška: $|0.14515 - 0.144599| = 0.55 \cdot 10^{-3}$

Zadatak 9 *Simpsonovom metodom s točnošću većom od 10^{-2} izračunajte $\int_0^1 \frac{dx}{1+3x}$. Odredite pravu grešku.*

Rješenje.

$$f(x) = \frac{1}{1+3x} \Rightarrow f'(x) = -\frac{3}{(1+3x)^2} \Rightarrow f''(x) = \frac{18}{(1+3x)^3} \Rightarrow f'''(x) = -\frac{162}{(1+3x)^4} \Rightarrow f^{iv}(x) = \frac{1944}{(1+3x)^5}$$

$$\Rightarrow M_4 = f(0) = 1944 \Rightarrow \frac{1}{180} \cdot 1944h^4 < 10^{-2} \Rightarrow 2n > 5.73 \Rightarrow 2n = 6.$$

x_i	$f(x_i)$
$x_0 = 0$	$f(x_0) = 1$
$x_1 = 1/6$	$f(x_1) = 0.66667$
$x_2 = 1/3$	$f(x_2) = 0.5$
$x_3 = 1/2$	$f(x_3) = 0.4$
$x_4 = 2/3$	$f(x_4) = 0.33333$
$x_5 = 5/6$	$f(x_5) = 0.28571$
$x_6 = 1$	$f(x_6) = 0.25$

$$\Rightarrow I_6 = \frac{1}{18}(1+4(0.66667+0.4+0.28571)+2(0.5+0.33333)+0.25) = 0.46256.$$

Kako je $\int_0^1 \frac{dx}{1+3x} = \left.\frac{\ln|1+3x|}{3}\right|_0^1 = 0.462098$, prava greška je $|0.462098 - 0.46256| = 0.467 \cdot 10^{-3}$.

Zadatak 10 *Koristeći Laplaceovu transformaciju odredite rješenje diferencijalne jednadžbe $x''(t) + x'(t) = \cos t - \sin t$ uz početne uvjete $x(0) = 1, x'(0) = 0$.*

Rješenje.

$$\mathcal{L}(x') = pX - x_0 = pX - 1, \;\; \mathcal{L}(x'') = p^2X - px_0 - x_0' = p^2X - p \Rightarrow p^2X - p + pX - 1 = \frac{p}{p^2+1} - \frac{1}{p^2+1}$$

$$\Rightarrow X = \frac{p^3 + p^2 + 2p}{p(p+1)(p^2+1)} = \frac{1}{p+1} + \frac{1}{p^2+1} \Rightarrow x(t) = e^{-t} + \sin t.$$

Zadatak 11 *Diferencijalnu jednadžbu $y' = -(1+x)^2,\; y(0) = 1$ na intervalu $[0, 0.2]$ s korakom $h = 0.1$ približno riješite Eulerovom metodom, te Runge-Kutta metodom i ocjenite koja je metoda točnija u točki $x = 0.1$ (izračunajte pravu grešku).*

Rješenje. Pravo rješenje:

$$dy = -(1+x)^2dx \Rightarrow y = -\frac{(1+x)^3}{3} + C \Rightarrow C = \frac{4}{3} \Rightarrow y = \frac{-(1+x)^3 + 4}{3} \Rightarrow y(0.1) = 0.88967.$$

Eulerova metoda:

$$y_1 = 1 + 0.1(-1) = 0.9 \Rightarrow y_2 = 0.9 + 0.1(-1.21) = 0.779.$$

Prava greška: $|0.9 - 0.88967| = 0.01033$.

Runge-Kutta metoda:

$$K_1^0 = -0.1, \quad K_2^0 = -0.11025, \quad K_3^0 = -0.11025, \quad K_4^0 = -0.121$$

$$\triangle y_0 = -0.11033 \Rightarrow y_1 = 0.88967$$

$$K_1^1 = -0.121, \quad K_2^1 = -0.13225, \quad K_3^1 = -0.13225, \quad K_4^1 = -0.144$$

$$\triangle y_1 = -0.13233 \Rightarrow y_2 = 0.75734$$

Prava greška: $|0.88967 - 0.88967| = 0$.

Točnija je Runge-Kutta metoda.

Zadatak 12 *Metodom zlatnog reza s greškom manjom od* $\varepsilon = 0.5$ *odredite minimum funkcije* $f(x) = e^x + \frac{7.8}{x}$ *na intervalu* $[1, 2]$.

Rješenje. Kako je $a^{(0)} = 1$ i $c^{(0)} = 2$ imamo

$$\frac{b^{(0)} - 1}{1} = \frac{3 - \sqrt{5}}{2} \Rightarrow b^{(0)} = 1.38197.$$

Kako je još

$$f(a^{(0)}) = f(1) = 10.51828, \quad f(c^{(0)}) = f(2) = 11.28906, \quad f(b^{(0)}) = 9.62686$$

$$\Rightarrow f(b^{(0)}) < f(a^{(0)}) \ \text{ i } f(b^{(0)}) < f(c^{(0)}),$$

početne su točke dobro odabrane.

Sada,

$$x^{(0)} = c^{(0)} + a^{(0)} - b^{(0)} = 1.61803, \quad f(x^{(0)}) = 9.8638 > f(b^{(0)})$$

$$\Rightarrow a^{(1)} = 1, \quad b^{(1)} = a^{(0)} + x^{(0)} - b^{(0)} = 1.23607, \quad c^{(1)} = 1.61803, \quad |c^{(1)} - a^{(1)}| = 0.61803 > 0.5$$

$$\Rightarrow x^{(1)} = c^{(1)} + a^{(1)} - b^{(1)} = 1.38197, \quad f(x^{(1)}) = 9.62686 < f(b^{(1)})$$

$$\Rightarrow a^{(2)} = 1.23607, \quad b^{(2)} = 1.38197, \quad c^{(2)} = 1.61803, \quad |c^{(2)} - a^{(2)}| = 0.38197 < 0.5$$

$$\Rightarrow x^* = (a^{(2)} + c^{(2)})/2 = 1.42705.$$