Zadatak 1 *S točnošću većom od 10^{-3} odredite $\sin 368°$. Izračunajte ukupnu grešku.* *(10)*

Rješenje.

$$\sin 368° = \sin 8° = \sin \frac{8\pi}{180} = \sin \frac{2\pi}{45}$$

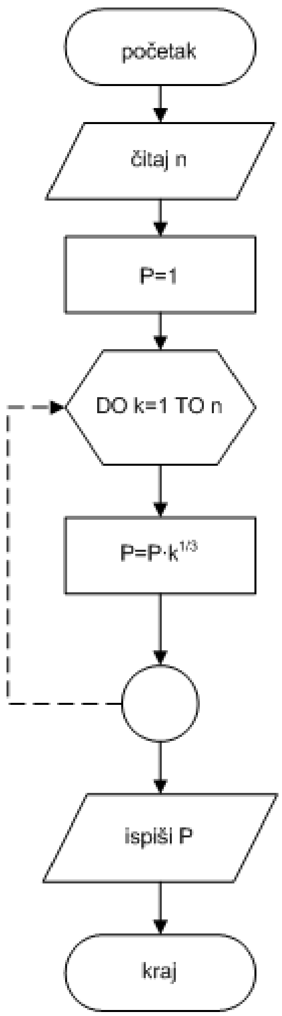
$$n = 2 \Rightarrow R_5\left(\frac{2\pi}{45}\right) \leq \frac{\left(\frac{2\pi}{45}\right)^5}{5!} = 0.442 \cdot 10^{-6} < 0.25 \cdot 10^{-3}$$

$$\Rightarrow \sin \frac{2\pi}{45} = \frac{2\pi}{45} - \frac{1}{6}\left(\frac{2\pi}{45}\right)^3 = 0.13963 - 0.00045 = 0.13918$$

$$\varepsilon = 0.442 \cdot 10^{-6} + 2 \cdot 0.5 \cdot 10^{-5} + 0 = 0.542 \cdot 10^{-4} < 10^{-3}.$$

Zadatak 2 *Opišite dijagram toka i napišite program u Mathematica-i za algoritam koji za zadani cijeli broj $n \geq 1$ (ulazna informacija) računa $\sqrt[3]{1} \cdot \sqrt[3]{2} \cdot \ldots \cdot \sqrt[3]{n}$.* *(15)*

Rješenje.




```
n = 10;
P = 1;
For[k = 1, k <= n, k = k + 1, P = P * ∛k];
Print[N[P]]
```

```
153.669
```

Slika 1:

Zadatak 3 *Gauss-Seidelovom metodom (jednom iteracijom) odredite približno rješenje sustava*

$$\begin{aligned} 4x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 4. \end{aligned}$$

Odredite pravu grešku. *(15)*

Rješenje.

$$L = \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\Rightarrow x^{(0)} = L^{-1}b = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow x^{(1)} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix} \left(\begin{bmatrix} 5 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{4} \\ \frac{1}{2} \end{bmatrix} \right) = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$$

Pravo rješenje:

$$\begin{aligned} 4x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 4 \end{aligned} \Leftrightarrow \begin{aligned} 4x_1 + 2x_2 &= 5 \\ -4x_1 - 6x_2 &= -8 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{7}{8} \\ \frac{3}{4} \end{bmatrix}$$

Prava greška:

$$\varepsilon = \sqrt{(0.875 - 1)^2 + (0.75 - 0.667)^2} = 0.150231$$

Zadatak 4 *Odredite vezu oblika* $\frac{a}{x^2} + by = ab$ *ako je* *(15)*

x_k	1	2	3
y_k	0.4	0.9	1

Rješenje.

$$by = ab - \frac{a}{x^2} \Rightarrow y = a - \frac{a}{b} \cdot \frac{1}{x^2} \Rightarrow \bar{y} = a_0 + a_1\bar{x}, \quad \bar{y} = y, \bar{x} = \frac{1}{x^2}, a_0 = a, a_1 = -\frac{a}{b}$$

$\bar{x}_i$	1	0.25	0.11
$\bar{y}_i$	0.4	0.9	1

$$\Rightarrow \sum_{i=0}^{2} \bar{x}_i = 1.36, \sum_{i=0}^{2} \bar{x}_i^2 = 1.0746, \sum_{i=0}^{2} \bar{x}_i\bar{y}_i = 0.735, \sum_{i=0}^{2} \bar{y}_i = 2.3 \Rightarrow a_0 = 1.07, a_1 = -0.67$$

$$\Rightarrow \ a = a_0 = 1.07, b = -\frac{a}{a_1} = 1.6 \Rightarrow \frac{1.07}{x^2} + 1.6y = 1.712$$

Zadatak 5 *Odredite polinom prvog stupnja koji u smislu metode najmanjih kvadrata najbolje aproksimira funkciju* $f(x) = 5^x$ *na intervalu* $[-1, 1]$. *(15)*

Rješenje.

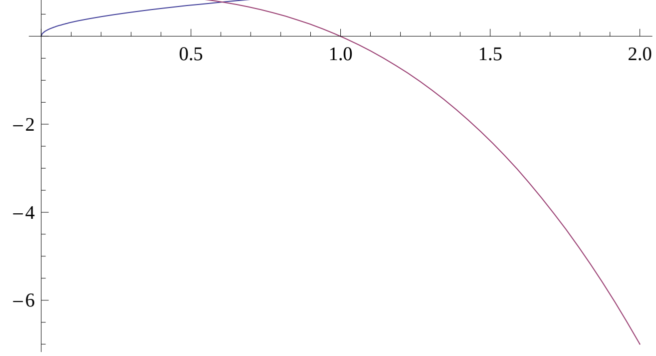
$$\int_{-1}^{1} x dx = 0, \ \int_{-1}^{1} x^2 dx = \frac{2}{3}, \ \int_{-1}^{1} 5^x dx = 2.98, \ \int_{-1}^{1} x \cdot 5^x dx = 1.38$$

$$a_0 = \frac{0.67 \cdot 2.98}{2 \cdot 0.67} = 1.49, \ a_1 = \frac{2 \cdot 1.38}{2 \cdot 0.67} = 2.06$$

$$\varphi(x) = 1.49 + 2.06x$$

Zadatak 6 *Za jednadžbu* $x^3 + \sqrt{x} = 1$ *odredite funkciju* φ *s kojom se može provesti metoda iteracije.* *(15)*

Rješenje.



Slika 2:

$$f(x) = 1 - x^3 - \sqrt{x}, \quad f(0) = 1 > 0, \quad f(1) = -1 < 0 \Rightarrow \text{ nultočka je unutar intervala } [0,1]$$

$$f'(x) = -3x^2 - \frac{1}{2\sqrt{x}} \Rightarrow M_1 = 3 + \frac{1}{2} = 3.5 \Rightarrow \lambda < \frac{2}{3.5} = 0.57 \Rightarrow \lambda = 0.5 \Rightarrow \varphi(x) = x - \frac{1}{2}(1 - x^3 - \sqrt{x})$$

Zadatak 7 *Newtonovom metodom (jednom iteracijom) odredite približno rješenje sustava* $x^2 + 20x + y^2 = 1, y = 0.5x + \sin xy$, *uzimajući za početne vrijednosti* $x_0 = y_0 = 0$. *(15)*

Rješenje.

$$F(X) = \begin{bmatrix} x^2 + 20x + y^2 - 1 \\ y - 0.5x - \sin xy \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow F'(X) = \begin{bmatrix} 2x + 20 & 2y \\ -0.5 - y\cos xy & 1 - x\cos xy \end{bmatrix} \Rightarrow [F'(x)]^{-1} = \frac{1}{\det[F'(X)]} \begin{bmatrix} 1 - x\cos xy & -2y \\ 0.5 + y\cos xy & 2x + 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{1}{20} \begin{bmatrix} 1 & 0 \\ 0.5 & 20 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.025 \end{bmatrix}$$

Zadatak 8 *Za funkciju* $f(x) = \ln 2x$ *poznate su vrijednosti* $f(0.5), f(1)$ *i* $f(1.5)$. *Odredite* $f'(1)$:

a) Hermiteovom metodom ako je još poznato i $f'(0.5)$, *(15)*

b) koristeći kubni splajn ako su poznate vrijednosti $f'(0.5)$ *i* $f'(1.5)$, *(15)*

c) numeričkim diferenciranjem. *(10)*

Izračunajte pravu grešku u sva tri slučaja.

Rješenje. a) $f'(x) = -3^{-x} \ln 3$

x_i	y_i	$f^{[1]}$	$f^{[2]}$	$f^{[3]}$
$x_{-1} = 1$	$y_{-1} = 0.69315$			
		$f'(x_{-1}) = ?$		
$x_{-1} = 1$	$y_{-1} = 0.69315$		$f[x_{-1}, x_{-1}, x_0] = ?$	
		$f[x_{-1}, x_0] = 1.3863$		$f[x_{-1}, x_{-1}, x_0, x_0] = ?$
$x_0 = 0.5$	$y_0 = 0$		$f[x_{-1}, x_0, x_0] = -1.2274$	
		$f'(x_0) = 2$		$f[x_{-1}, x_0, x_0, x_1] = 0.65202$
$x_0 = 0.5$	$y_0 = 0$		$f[x_0, x_0, x_1] = -0.90139$	
		$f[x_0, x_1] = 1.09861$		
$x_1 = 1.5$	$y_1 = 1.09861$			

$$\frac{-1.2274 - f[x_{-1}, x_{-1}, x_0]}{-0.5} = 0.65202 \Rightarrow f[x_{-1}, x_{-1}, x_0] = -0.90139$$

$$\Rightarrow \frac{1.3863 - f'(x_{-1})}{-0.5} = -0.90139 \Rightarrow f'(1) = 0.93561.$$

Kako je prava vrijednost $f'(1) = 1$ za pravu grešku imamo $|1 - 0.93561| = 0.06439$.

b)

x_i	y_i	$f[x_i, x_{i+1}]$
$x_0 = 0.5$	$y_0 = 0$	
		$f[x_0, x_1] = 1.3863$
$x_1 = 1$	$y_1 = 0.69315$	
		$f[x_1, x_2] = 0.81092$
$x_2 = 1.5$	$y_2 = 1.09861$	

$$\Rightarrow 0.5s_0 + 2s_1 + 0.5s_2 = 3 \cdot 0.5 \cdot 2.19722 = 3.29583.$$

Kako je i $f'(0) = 2$ i $f'(1.5) = 0.66667$, imamo $s_1 = 0.98125$, a prava grška je $|1-0.98125| = 0.01875$.

c)

$$f'(1) = \frac{1}{2 \cdot 0.5}(f(1.5) - f(0.5)) = 1.09861.$$

Greška: $|1 - 1.09861| = 0.09861$.

Zadatak 9 *Simpsonovom metodom s točnošću većom od 10^{-6} izračunajte $\int_1^2 \sqrt[3]{2+x}dx$. Odredite pravu grešku.*

(15)

Rješenje.

$$f(x) = \sqrt[3]{2+x} \Rightarrow f'(x) = \frac{1}{3}(2+x)^{-\frac{2}{3}} \Rightarrow f''(x) = -\frac{2}{9}(2+x)^{-\frac{5}{3}} \Rightarrow f'''(x) = \frac{10}{27}(2+x)^{-\frac{8}{3}} \Rightarrow f^{iv}(x) = -\frac{80}{81}(2+x)^{-\frac{11}{3}}$$

$$\Rightarrow M_4 = -f(1) = 0.0176 \Rightarrow \frac{h^4}{180} \cdot 0.0176 < 10^{-6} \Rightarrow 2n > 3.14456 \Rightarrow 2n = 4.$$

x_i	$f(x_i)$
$x_0 = 1$	$f(x_0) = 1.4422496$
$x_1 = 1.25$	$f(x_1) = 1.481248$
$x_2 = 1.5$	$f(x_2) = 1.5182945$
$x_3 = 1.75$	$f(x_3) = 1.5536162$
$x_4 = 2$	$f(x_4) = 1.587401$

$$\Rightarrow I_4 = \frac{1}{12}(1+4(1.481248+1.5536162)+2\cdot1.5182945+1.587401) = 1.5171414.$$

Kako je $\int_1^2 \sqrt[3]{2+x}dx = \frac{3}{4}(2+x)^{\frac{4}{3}}\Big|_1^2 = 1.5171417$, prava greška je $|1.5171414-1.5171417| = 0.3\cdot10^{-6}$.

Zadatak 10 *Koristeći Laplaceovu transformaciju odredite rješenje diferencijalne jednadžbe $x''(t) - 4x'(t) - 5x(t) = -10$ uz početne uvjete $x(0) = 3, x'(0) = 5$.* *(15)*

Rješenje.

$$\mathcal{L}(x') = pX-x_0 = pX-3,\ \mathcal{L}(x'') = p^2X-px_0-x_0' = p^2X-3p-5 \Rightarrow p^2X-3p-5-4pX+12-5X = -\frac{10}{p}$$

$$\Rightarrow X = \frac{3p^2-7p-10}{p(p^2-4p-5)} = \frac{2}{p} + \frac{1}{p-5}$$

$$\Rightarrow x(t) = 2 + e^{5t}.$$

Zadatak 11 *Diferencijalnu jednadžbu $y' = \frac{x+1}{y}$, $y(0) = 2$ na intervalu $[0,1]$ s korakom $h = 0.5$ približno riješite Eulerovom metodom, te Runge-Kutta metodom i ocjenite koja je metoda točnija u točki $x = 0.5$ (izračunajte pravu grešku).* *(15)*

Rješenje. Pravo rješenje:

$$ydy = (x+1)dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + x + C \Rightarrow C = 2 \Rightarrow y = \sqrt{x^2+2x+4} \Rightarrow y(0.5) = 2.29129.$$

Eulerova metoda:

$$y_1 = 2 + 0.5 \cdot \frac{1}{2} = 2.25$$

$$y_2 = 2.25 + 0.5 \cdot \frac{1.5}{2.25} = 2.58333$$

Prava greška: $|2.25 - 2.29129| = 0.04129$.

Runge-Kuttina metoda:

$$K_1^0 = 0.25, \quad K_2^0 = 0.29412, \quad K_3^0 = 0.2911, \quad K_4^0 = 0.32735$$

$$\triangle y_0 = 0.291298 \Rightarrow y_1 = 2.291298$$

$$K_1^1 = 0.32732, \quad K_2^1 = 0.35642, \quad K_3^1 = 0.35432, \quad K_4^1 = 0.37798$$

$$\triangle y_1 = 0.35446 \Rightarrow y_2 = 2.64576$$

Prava greška: $|2.291298 - 2.29129| = 0.8 \cdot 10^{-5}$.

Točnija je Runge-Kuttina metoda.

Zadatak 12 *Koristeći shemu konačnih razlika približno rješite rubni problem za parcijalnu diferencijalnu jednadžbu prvog reda:*

$$\begin{cases} \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = x \cdot y, & na \quad S = [0,1] \times [0,1] \\ u(x,y) = x + y, & na \quad \Gamma = \partial S \end{cases}$$

$s\ h = k = 0.5.$ *(15)*

Rješenje.

$$\frac{u_{ij+1} - u_{ij}}{h} + \frac{u_{i+1j} - u_{ij}}{h} = x_i \cdot y_j \Rightarrow u_{ij+1} + u_{i+1j} - 2u_{ij} = 0.5 x_i y_j.$$

Kako je

$$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1, \quad y_0 = 0,\ y_1 = 0.5,\ y_2 = 1,$$

zbog rubnog uvjeta imamo

$$u_{00} = 0,\ u_{01} = u_{10} = 0.5,\ u_{02} = u_{20} = 1,\ u_{12} = u_{21} = 1.5,\ u_{22} = 2.$$

Sada, za $i = j = 1$ imamo

$$u_{12} + u_{21} - 2u_{11} = 0.5 \cdot 0.5 \cdot 0.5 \Rightarrow u_{11} = 1.4375.$$